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A comprehensive Analysis of Degree Based Condition for Hamiltonian Cycles

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***Abstract* — Rahman and Kaykobad introduced a shortest distance based condition for finding the existence of Hamiltonian paths in graphs as follows: Let G be a connected graph with n vertices, and if $d(u) + d(v) + \delta(u,v) \geq n+1$, for each pair of distinct non-adjacent vertices u and v in G, where $\delta(u,v)$ is the length of a shortest path between u and v, then G has Hamiltonian path. Rao Li proved that under the same condition, the graph is Hamiltonian or belongs to two different classes of graphs. Recently, Mehedy, Hasan and Kaykobad showed case by case that under the condition of Rahman and Kaykobad, the graph is Hamiltonian with exceptions for $\delta(u,v)$ =2. Shengjia Li et. al. mentions a graph to be Hamiltonian whenever $d(u)+d(v) \geq n-1$, for all $\delta(u,v)=2$, otherwise n is odd and the graph falls into a special class. This paper relates the results of Mehedy, Hasan and Kaykobad with the two exceptional classes of graphs introduced by Rao Li and the graph class introduced by Shengjia Li et. al. The paper also provides a thorough analysis of the graph classes and shows the characteristics of a graph when it falls into one of those classes.**

***Index Terms* — Hamiltonian cycle, Hamiltonian path, Graphs.**

I. INTRODUCTION

We consider only simple undirected graphs- graphs that do not contain loops or multiple edges. A Hamiltonian cycle is a closed path passing through every vertex of a graph. A graph containing a Hamiltonian cycle is said to be simply Hamiltonian. By Hamiltonicity we mean the virtue of a graph to be Hamiltonian. Naturally every Hamiltonian graph contains a Hamiltonian path and not necessarily vice versa. We define $\delta(u,v)$ as the shortest distance between u and v, K_p as the complete graph with p vertices, K_p^c complement of K_p(a set of p independent vertices), $K_{p,q}$ as complete bipartite graph with p + q vertices. '$\vee$' denotes join operator. We term the condition: "$d(u)+d(v)+\delta(u,v) \geq n+1$ *,where* $|V|=n$ *and* u,v are distinct non-adjacent vertices of G ", as Rahman and Kaykobad condition.

We also describe three families of graph C_n, D_n and L_n, defined in [4],[4],[5] respectively.

C_n is defined as $\{G: V(G) = V(K_1) \cup V(G_1) \cup V(G_2)$, $E(G)=E(G_1) \cup E(G_2) \cup \{ab:$ *where* $a \in V(K_1)$ and $b \in V(K_{p1}) \cup V(K_{p2})\}$, where K_1 consisting of one single vertex, $G_1 = K_{p1} \vee K_{q1}$ is a complete graph with p_1+q_1 vertices such that $p_1 \geq 1$ and $q_1 \geq 0$, $G_2 = K_{p2} \vee K_{q2}$ is a complete graph with p_2+q_2 vertices such that $p_2 \geq 1$ and $q_2 \geq 0$ and $V(K_1), V(G_1)$ and $V(G_2)$ are pair wise disjoint$\}$

D_n is defined as $\{G: K_{p,p+1} \subseteq G \subseteq K_p \vee (p+1)K_1$, where $|V(G)|=2p+1 \geq 3\}$, where $(p+1)K_1$ denotes (p+1) isolated vertices.

L_n is defined as $L_n = L_{2m+1} = \{Z_m \vee (K_m^c + \{u\}) \mid Z_m$ is a graph with m vertices$\}$, where K_m^c is a set of m independent vertices (also as the complement of the complete graph K_m) and u is another single vertex, furthermore, the edge set of L_n consists of $E(Z_m)$ and $xy|x \in V(Z_m)$ and $y \in K_m^c \cup \{u\}\}$

It is to be noted that D_n and L_n are actually the same class of graphs with different names. We adapt the notation D_n in our results to represent both the classes.

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II. EXISTING RESULTS

To prove our results, we need some existing theorems. We list them in this section for the same of completeness of this article.

Theorem 1.1 (Rahman and Kaykobad [3]). *Let* $G=(V,E)$ *be a connected graph with n vertices such that for all pairs of distinct non-adjacent vertices* $u,v\in V$ *, we have* $d(u)+d(v)+\delta(u,v)\geq n+1$ *. Then G has a Hamiltonian path.*

Theorem 1.2 (Rao Li [4]). *Let* $G=(V,E)$ *be a connected graph with n vertices. If* $d(u)+d(v)+\delta(u,v)\geq n+1$ *for each pair of distinct non-adjacent vertices* $u,v\in V$ *, then G is Hamiltonian or* $G\in C_n\cup D_n$

Theorem 1.3 (Mehedy, Hasan, and Kaykobad [1]). *Let* $G=(V,E)$ *be a* ***2-connected*** *graph with n vertices such that for all pairs of distinct non-adjacent vertices* $u,v\in V$ *we have* $d(u)+d(v)+\delta(u,v)\geq n+1$ *then*

i) $\delta(u,v)\leq 3$ *, for all* $u,v\in V$

ii) If for any $u,v\in V$ $\delta(u,v)$ *=3, G is Hamiltonian*

iii) If $\delta(u,v)$ *=2 for all non-adjacent vertices, G may or may not be Hamiltonian. To make the graph Hamiltonian, it needs only one extra edge added to the graph.*

Theorem 1.3 is cumulative results of the work [1], to help our analysis.

Theorem 1.4 (Shengjia Li et. al.[5]). *Let G be a 2-connected graph with* $n\geq 3$ *vertices. If* $d(u)+d(v)\geq n-1$ *for every pair of vertices u and v with d(u,v)=2, then G is Hamiltonian, unless n is odd and* $G\in L_n$.

Theorem 1.5 (Ore [2]). *If* $d(u)+d(v)\geq n$ *for every pair of distinct non-adjacent vertices u and v of G, then G is Hamiltonian.*

We take Theorem 1.3 as the base and fulfill the missing results in the theorem by using other theorems above. We also provide a through analysis of the graph classes and show the characteristics of a graph when it falls into one of those classes.

III. MAIN RESULTS

This paper relates the results of Mehedy, Hasan and Kaykobad [1] with the two exceptional classes of graphs introduced by Rao Li [4] and the graph class introduced by Shengjia Li et. al. [5] and brings the results of all three works under Rahman and Kaykobad condition. We analyze when a graph falls in the two classes C_n and D_n and what the characteristics of a graph instance are. We provide a unified theorem as follows:

Theorem 2.1. *Let* $G=(V,E)$ *be a* ***connected*** *graph with n vertices such that for all pairs of distinct non-adjacent vertices* $u,v\in V$ *we have* $d(u)+d(v)+\delta(u,v)\geq n+1$ *then*

i) $\delta(u,v)\leq 4$ *, for all* $u,v\in V$

ii) If there are vertices $u,v\in V$ *with* $\delta(u,v)$ *=4, then* $G\in C_n$

iii)If there are vertices $u,v\in V$ *with* $\delta(u,v)$ *=3 and no other vertices* $u',v'\in V$ *with* $\delta(u',v')$ *>3, then G is Hamiltonian if 2-connected or else* $G\in C_n$

iv) If $\delta(u,v)$ *=2 for all non-adjacent vertices, G Hamiltonian or* $G\in D_n$, given n=odd*.* If G is not Hamiltonian, then adding just one extra edge makes the graph Hamiltonian.

Proof: Theorem 1.3 along with Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4 proves the theorem 2.1.

Lemma 2.1. *If* $d(u)+d(v)+\delta(u,v)\geq n+1$*, then for all pair of distinct non-adjacent vertices* $u,v\in V$ *,* $\delta(u,v)\leq 4$

Lemma 2.2. *If* $d(u)+d(v)+\delta(u,v)\geq n+1$*, and there are vertices* $u,v\in V$ *with* $\delta(u,v)$ =4, then $G\in C_n$

Lemma 2.3. *If* $d(u)+d(v)+\delta(u,v)\geq n+1$ *and* for any $u,v\in V$ $\delta(u,v)$ =3 and no other vertices $u',v'\in V$ with $\delta(u',v')$ >3 and G is not 2-connected, then $G\in C_n$

Lemma 2.4. *If* $d(u)+d(v)+\delta(u,v)\geq n+1$ *and* for every non-adjacent pair $u,v\in V$ $\delta(u,v)$=2 and n=odd, then $G\in D_n$.

Proof of Lemma 2.1: Theorem 1.3[1] proves that under Rahman and Kaykobad condition $\delta(u,v)\leq 3$ when G is 2-connected. On the other hand Theorem 1.2[4] proves that under Rahman and Kaykobad condition $G \in C_n \cup D_n$ where G is 2-connected or just connected.

Now lets find out the maximum possible $\delta(u,v)$ *in* C_n and D_n. *In* C_n maximum $\delta(u,v)$ *=4* is possible only when $u \in K_{q1}$ and $v \in K_{q2}$. *In* D_n maximum $\delta(u,v)$ *=2*.

So, under Rahman and Kaykobad condition, $\delta(u,v) \le 3$, when G is 2-connected and $\delta(u,v)$ *=4,* when G is not two connected. In general, $\delta(u,v)$ *=4,* under Rahman and Kaykobad condition.

Proof of Lemma 2.2: In the proof of Lemma 2.1 section we discussed that under Rahman and Kaykobad condition, $\delta(u,v)$ *=4,* only when G is not 2-connected. If a graph is not two connected, it cannot be Hamiltonian. So, according to Theorem 1.2[4], the graph falls into C_n or D_n. But as $\delta(u,v)$ *=4,* the graph cannot fall in D_n and hence falls into the category C_n.

Proof of Lemma 2.3: We prove the lemma using the same arguments Rao Li [4] used to prove his theorem.

Let P: =us$_1$s$_2$v be the shortest path between u and v with length 3. So there exists at least a single pair of distinct non-adjacent vertices in G (example u and s_2 or s_1 and v) such that the shortest distance between them is 2.

Since G is not 2-connected (has cut vertex), there exist a vertex say, x in G such that G[V(G)\{x}] is disconnected. Let G_i be the components in G[V(G)\{x}] and the number of vertices in G_i is n_i, where $2 \le l$ and $1 \le i \le l$.

So, there must be two vertices in two different components $u_i \in V(G_i)$, such that $u_i x \in E(G)$, where i=1, 2 and $\delta(u_1 u_2) = 2$. Now,

$$n+1 \le d(u_1) + d(u_2) + 2 \le n_1 + n_2 + 2$$
$$\le n_1 + n_2 + n_3 + \ldots + n_l + 2 = n+1$$

So, $d(u_1)=n_1$, $d(u_2)=n_2$ and $n_3=n_4=\ldots\ldots=n_l=0$. So, there will be only two components and u_i will be connected to all the vertices in $\mathrm{V(G_i)} \cup \{\mathrm{x}\}$, i=1, 2 and $n=n_1+n_2+1$. Now let's, define the four set of graphs present in C_n:

$$U_1 = \{u \in V(G_1) : ux \in E\}, |U_1| \ne 0$$
$$V_1 = \{v \in V(G_1) : vx \notin E\}, |V_1| \ge 0$$
$$U_2 = \{u \in V(G_2) : ux \in E\}, |U_2| \ne 0$$
$$V_2 = \{v \in V(G_2) : vx \notin E\}, |V_2| \ge 0$$

Any vertex u' other than u_1 in the set U_1 will show $\delta(u', u_2)$=2 and $d(u') = n_1$. So u' is adjacent to all the vertices in $\mathrm{V(G_1)} \cup \{\mathrm{x}\}$. Symmetrically, it can be shown that any vertex u'' in U_2 is adjacent to all vertices in $\mathrm{V(G_2)} \cup \{\mathrm{x}\}$.

Let v' be any vertex in V_1 and so $\delta(v', u_2)$ =3. $n+1 \le d(v') + d(u_2) + 3 \le n_1 - 1 + n_2 + 3 = n+1$, which indicates $d(v') = n_1 - 1$ and v' is exactly adjacent to all the vertices in $V(G_1)$. Similarly any vertex v'' in V_2 is adjacent to all the vertices in $V(G_2)$. Hence $\mathrm{G} \in \mathrm{C_n}$.

The lemma can be proved by the discussion in Lemma 2.1 also. As G is not 2-connected, it is not Hamiltonian. So, according to Theorem 1.2[4], $G \in C_n \cup D_n$. But as maximum $\delta(u,v)$ *=2* in D_n, G is actually of class C_n.

Now we analyze the characteristics of $\mathrm{G} \in \mathrm{C_n}$. For any non-adjacent vertex pair $u, v \in E(G)$, $\delta(u,v) = 3$ implies that no vertex $w \in V$ can be adjacent to u, v at the same time since then $\langle u \rightarrow w \rightarrow v \rangle$ would have been a path of length 2. $\delta(u,v) = 3$, also implies $d(u) + d(v) \ge n-2$. As, u and v cannot have a common neighbor, and the number of vertices that can be connected is (n-2). Hence, by contradiction, $d(u) + d(v) = n-2$ is proved. So, u, v connects to all the vertices in the graph, having no common neighbor.

As we assume 1-connected graph (Fig.1.), there should be no cross over edges like $(1, i+1), (i, n) \in E$. Because cross over edge will make the graph 2-connected, and also Hamiltonian. In Fig.1, we see that there is two cut vertices k and (k+1).

We know that under Rahman and Kaykobad condition $\delta(u,v) \le 3$ for all $u, v \in V$. To maintain this, vertices from u=1 to (k-1), all should be connected to k, so than $\delta(i, v=n) = 3$, $i \in \{1 \ \ to \ \ k-1\}$. Similarly vertices from (k+2) to v=n should be connected to (k+1) to maintain $\delta(i, v=1) = 3$, $i \in \{k+2 \ \ to \ \ n\}$. Now, the non adjacent vertices on the sides {1 to k} and {k+1, to n}, have a distance 2. So, any two non adjacent vertices { $u', v' \in$ (1 to k-1) or (k+2 to n) } should maintain $d(u') + d(v') \ge n-1$. But maximum possible $d(u') + d(v') = n-2$, only when k=2 or k=n-1. Hence to maintain Rahman Kaykobad condition, the graph component composed of vertices {1 to k} will be complete and similarly the component with vertices {k+1 to n} is complete. So, the graph falls in C_n.

We define this class of graph as $C_n' \subset C_n$, where { $V(K_{p1}) = \{k\}$, $V(K_{q1}) = \{1 \ \ to \ \ k-1\}$,

$V(K_1) = \{k+1\}$, $V(K_{p2}) = \{k+1 \;\; to \;\; n\}$, $V(K_{q2}) = \{\phi\}$
and $k \in \{2 \;\; to \;\; n-1\}$ }

Proof of Lemma 2.4: It is to be noted that Rahman and Kaykobad condition fulfills the condition given in theorem satisfying the above condition is: $|V(Z_p)|=p$ and $|(p+1)K_1|=p+1$, n=2p+1

In this setup, adding just one edge in (p+1) K1 makes the graph Hamiltonian, which is a finding of work[1]. It is to be noted that adding that edge reduces the independent set size by two, and becomes p-1 whereas $|V(Z_p)|$ becomes p+2 ($|(p+1)K_1| \leq |V(Z_p)|$), which implies Hamiltonicity in the graph.

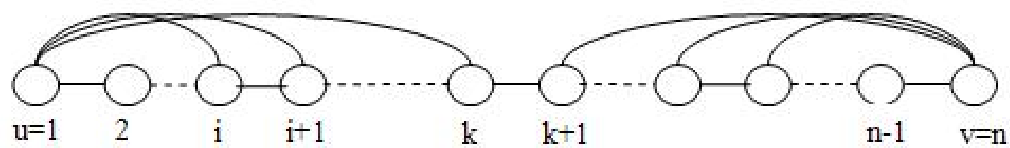

Fig.1. An 1-connected graph G fulfilling Rahman and Kaykobad condition, with $\delta(u,v) = 3$

1.4 [5]**.** In Rahman and Kaykobad, for any pair $u,v \in V$ *if* $\delta(u,v)$ =2, $d(u)+d(v) \geq n-1$. If for all non-adjacent pair $u,v \in V$ $\delta(u,v)$ =2, Rahman and Kaykobad condition ensures 2-connectedness. So, the Lemma can directly be derived from Theorem 1.4.

Part (iii) of Theorem 1.3 [1] misses one fact that G is Hamiltonian when n=even. Lets see why n=even makes the graph Hamiltonian. We define a graph class D_n' $\{G : K_{p,q} \subseteq G \subseteq K_p \vee qK_1$, where $|V(G)| = even$ }. D_n' is similar to class D_n, with the exceptions that n=even and hence $q \neq p+1$. Note that any graph G with $|V(G)| = even$, falls into this class.

To keep $\delta(u,v)$=2 for all non-adjacent pair $u,v \in V$, the graph G∈ D_n' should provide an independent set {V (qK_1)}, to which all other vertices {V ($Z_p \subseteq K_p, |V(Z_p)| = p$)} should be connected. The independent set should be $> n/2$ to avoid Hamiltonicity, but doing so breaks the Rahman Kaykobad conditions (one can verify the fact easily by drawing a graph). So $q \leq p$, which eventually makes the graph Hamiltonian:
$\{ u \in V(Z_p)s_1 \in V(qK_1)s_2 \in Z_p s_3 \in V(qK_1)\text{.........} v \in Z_p.a \in V(qK_1)u, \delta(u,v) = 2 \}$.

Now for n=odd, the number of independent vertices should be kept larger than the number of vertices in Z_p and at the same time the Rahman and Kaykobad condition should be satisfied. The only possible configuration

Corollary 2.1. C_n and D_n have Hamilton-path.

Graph classes C_n and D_n fulfills the Rahman and Kaykobad condition but are not Hamiltonian. So, according to Theorem 1.1 [3], they both have Hamilton-path.

Corollary 2.2. *Under the condition of Rahman and Kaykobad, if every non-adjacent* $u,v \in V$ $\delta(u,v)$ *=2, then the graph is Hamiltonian or the size of the independent set is* $\left\lfloor \frac{n}{2} \right\rfloor$

Proof: To avoid Hamiltonicity The size of the independent set should be $> n/2$. However, doing so breaks the Rahman Kaykobad conditions, when n=even. So, the graph is Hamiltonian when n=even. This is also proven in Theorem 1.4[5] in a different way.

When n=odd, it is possible to maintain Rahman and Kaykobad condition even making size of the independent set $> n/2$ only when the independent set size is $\frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil$. This can also be obtained by examining the graph family D_n found in [4],[5].

Hence the corollary is proved. We also generalize two corollaries derived by rao Li[4].

Corollary 1.1 (Rao Li[4])**.** *Let G be a 2-connected graph with n vertices. If* $d(u)+d(v)+\delta(u,v) \geq n+2$ *for each pair of distinct non-adjacent vertices* u *and* v *in G, then G is pancyclic or* $G \in \{K_{n/2,n/2} : \;\; where \;\; n \geq 4 \;\; is \;\; even\}$.

Corollary 2.3. *Given the condition in Corollary 1.1, G is Hamiltonian.*

Proof: If for all $u, v \in V$ $\delta(u,v)$ =2, the condition given reduces to $d(u)+d(v) \geq n$ which is actually Ore's condition [2], implying Hamiltonicity (see theorem 1.5). If for any $u, v \in V$ $\delta(u,v)$ =3, the condition becomes $d(u)+d(v) \geq n-1$. However, Mehedy, Hasan and Kaykobad [1] (see theorem 1.3) proved that for $\delta(u,v)$ =3, with $d(u)+d(v) \geq n+1-3=n-2$, the graph becomes Hamiltonian. It is also proved in theorem 1.3 [1] that $\delta(u,v)$ cannot be greater than 3 for $d(u)+d(v)+\delta(u,v) \geq n+1$. $d(u)+d(v)+\delta(u,v) \geq n+2$ is a relaxed condition allowing more edges to be present in the graph and hence the finding $\delta(u,v) \leq 3$, is also applicable here. So, we say that Corollary 1.1 can be more generalized and the condition given in Corollary 1.1 proves the graph to be Hamiltonian.

Corollary 1.2 (Rao Li[4])**.** *Let G be a 3-connected graph with n vertices. If* $d(u)+d(v)+\delta(u,v) \geq n+3$ *for each pair of distinct non-adjacent vertices* u *and* v *in G, then G is Hamilton-connected.*

Corollary 2.4. *Given the condition in Corollary 1.2, G is Hamiltonian.*

Proof: The same arguments provided for Collorary 2.3 applies here also. It is to be noted that for $\delta(u,v)$=2, the condition is: $d(u)+d(v) \geq n+1$, which is more relaxed than Ore's condition[2]. We also argue that $\delta(u,v) \leq 3$ and for $\delta(u,v)$ =3, the graph is Hamiltonian by using theorem 1.3 [1] and because the graph is 3-connected (2-connected graph is more constrained by number of edges than 3 connected). So, the graph is Hamiltonian.

IV. CONCLUSION

We relate three seminal works for finding the Hamiltonicity of a graph observing Rahman and Kaykobad condition or a subset of the condition. We determine when the graph is Hamiltonian. If we cannot, we characterize the non-Hamiltonian graph by using two families of graphs and which instances of graphs in those two families, the graph falls in. Our work provides a comprehensive understanding of graphs following Rahman and Kaykobad condition.

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